In the paper some issues related to the control of torsional vibrations in a multi-mass system are presented. In the introduction a short comparison between different estimators is presented. Then the mathematical model of the three-mass system is introduced. Next the structure of the Luenberger observer is presented. The design methodology of this estimator for a three-inertia system is described in detail. Next a robustness analysis of the Luenberger observer for the system with measurement noises and parameter changes is shown. The included simulation results confirm the theoretical analysis.

1. INTRODUCTION

Multi-mass drive systems play an important role in many modern industrial drive applications. In such systems, the load (machine) is connected to a driving motor through a mechanical part including shafts and gearboxes. During transient operation, inertias of the motor-gearbox-load system and shafts elasticity cause the load speed to differ from the motor speed. This motor-gearbox-load speed imbalance greatly influences the torque transmission properties of the drive system and results in increased angular vibrations of the shafts. Excessive shafts twists and poorly damped torsional vibrations are detrimental to the drives performance greatly compromising product quality and system reliability, and in some cases leading to instability and failure of the entire drive system. This problem commonly occurs in rolling-mill drives, conveyer drives, robotic-arm drives including space manipulators, servo-drives and throttle systems [1]–[5].

In order to damp the torsional vibrations effectively, many control structures have been developed. The most advanced systems are based on the feeding back to the control structures the additional feedbacks from all (or selected) state variables. As an examples can serve the papers where the different possibilities of the modification of

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the cascade control structures [5] or [6] where the state controller structure is described. However, in practical application usually the signal of the electromagnetic torque and the motor speed are available for the control purposes. Therefore, special systems for estimation of those variables are necessary.

In many papers the Luenberger observers (LO) have been applied to the non-measurable state variables reconstruction [6], [7]. The main advantages of this estimator is the simplicity of the design and low computational requirements. In the case of the system with constant value of the measurements and process noises this estimator with suitable selected gain matrix can ensure the proper performances of the drive system. For the system with changeable but know value of the noises the Kalman filter (KF) is claim to be the optimal estimator [8], [9]. It comes from the fact that KF algorithm relies directly on the algorithm which minimize the effect of the parameter and measurement noises. However, this estimator possesses also some drawbacks. It is assumed that the noises have to be white and have Gaussian distribution which is not always true in the case of the modern drive system. The other drawbacks of KF is a complicated algorithm which causes quite a long computational time. It means that practical implementation of KF algorithm requires faster processor than Luenberger observer. The next difficulty is related to proper setting of the covariance matrices $Q$ and $R$, which determine dynamical characteristics of this estimator. Those shortcomings limit the wide application of the KF in industry.

In the analysis of the drive systems with flexible couplings different mathematical models can be used [10]. Usually the inertia-shaft-free two-mass model of the drive with the elastic coupling is considered. However, in the case of the system with mechanical gear the two-inertia model is not accurate enough in high-performance application. This problem is especially important in the case of a drive system working in the closed-loop structure with required high performance dynamics.

The main goal of the paper is the analysis of LO for the three-inertia system. First the mathematical model of the considered drive system is presented. Then the way of design of the LO is presented. Then the analysis of the dynamic performance of LO is presented. The theoretical consideration are supported by the simulation study

2. THE MATHEMATICAL MODEL OF THE DRIVE

In technical papers many mathematical models, which can be used for the analysis of the plant with elastic coupling exist. In many cases the drive system can be modeled as a two-mass system. However, in the application with the gearbox this model cannot ensure the accurate description of the real system, especially when the inertia of the gearbox is relatively big. In this cases the three-inertia model should be used to analysis and the design of the control structure of the drive system. In this model the first
mass represents the moment of inertia of the driving motor, the second mass refers to the moment of gearbox and the third mass represents the inertia of the load side. The mechanical couplings (between motor and gearbox as well as the gearbox and the load machine) are treated as inertia free. The schematic diagram of the three-mass system is presented in Fig. 1.

![Schematic diagram of the three-mass system](image)

**Fig. 1. The schematic diagram of the three-mass system**

The analyzed system is described by the following state equation (in per unit system):

\[
\frac{dx(t)}{dt} = Ax(t) + Bu(t) + Dz(t),
\]

\[
y(t) = Cx(t)
\]

where the system vectors and matrices are defined as follows:

\[
u = [m_e], \quad x = [\omega_1 \omega_2 \omega_3 \ m_{s1} \ m_{s2}]^T, \quad z = [m_L], \quad (2)
\]

\[
A = \begin{bmatrix}
0 & 0 & 0 & -1/T_1 & 0 \\
0 & 0 & 0 & 1/T_2 & -1/T_2 \\
0 & 0 & 0 & 0 & 1/T_3 \\
1/T_{c1} & -1/T_{c1} & 0 & 0 & 0 \\
0 & 1/T_{c2} & -1/T_{c2} & 0 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
1/T_1 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}, \quad C = \begin{bmatrix}
1^T \\
0 \\
0 \\
0 \\
0
\end{bmatrix}; \quad D = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix} \quad (3)
\]

where: \(\omega_1, \omega_2, \omega_3\) – the motor, gearbox and load machine speeds, \(m_{s1}, m_{s2}\), shaft torques in the beam between the motor and the gearbox as well as between the gearbox and the load machine, \(T_1, T_2, T_3\) – mechanical time constant of the motor, gearbox and the load machine, \(T_{c1}, T_{c2}\) – stiffness time constant of the first and the second couplings.
3. THE LUENBERGER OBSERVER

For the continuous linear system the Luenberger observer is described by the following state equation:

\[
\frac{d}{dt} x_{Re}(t) = A_R x_{Re}(t) + B_R u_R(t) + K(y - y_e),
\]

\[
y_e(t) = C_R x_R(t)
\]

where: \( K \) – is the gain matrix of the observer, \( y_e \) is the output signal of the observer, \( x_{Re} \) is the estimated state vector of the system, \( u_R \) is the control signal.

The original state vector is extended by the load torque. Such extended state vector is described by the following equation:

\[
x_{Re} = [\omega_{le} \ \omega_{2e} \ \omega_{3e} \ m_{s1e} \ m_{s2e} \ m_{Le}]^T.
\]

The system matrices are defined as follows:

\[
A_R = \begin{bmatrix}
0 & 0 & 0 & -\frac{1}{T_1} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{T_2} & -\frac{1}{T_2} & 0 \\
0 & 0 & 0 & 0 & \frac{1}{T_3} & -\frac{1}{T_3} \\
\frac{1}{T_{c1}} & -\frac{1}{T_{c1}} & 0 & 0 & 0 & 0 \\
0 & \frac{1}{T_{c2}} & -\frac{1}{T_{c2}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, \\
B_R = \begin{bmatrix}
\frac{1}{T_1} \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}; \\
C_R = \begin{bmatrix}
1 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}.
\]

The input signal is the electromagnetic torque of the driving motor and the output signal is the motor speed.

With the help of the following equation

\[
p(s) = \det(sI - (A - K \cdot C)),
\]

the characteristic equation of the system is obtained:

\[
\begin{align*}
& s^6 + s^5 \frac{k_1}{T_1} + s^4 \frac{T_1 T_2 T_{c1} + T_1 T_2 T_{c2} + T_1 T_3 T_{c1} + T_2 T_3 T_{c2} - T_2 T_3 T_{c2} k_2}{T_1 T_2 T_3 T_{c1} T_{c2}} + \\
& + s^3 \frac{k_3 (T_2 T_{c1} + T_3 T_{c1} + T_2 T_{c2}) + T_3 T_{c2} k_3}{T_1 T_2 T_3 T_{c1} T_{c2}} + \\
& + s^2 \frac{T_1 + T_2 + T_3 - k_2 (T_2 + T_3) - T_3 k_4}{T_1 T_2 T_3 T_{c1} T_{c2}} + s k_1 + k_3 + k_5 + s \frac{1}{T_1 T_2 T_3 T_{c1} T_{c2}} + \frac{-k_6}{T_1 T_2 T_3 T_{c1} T_{c2}}.
\end{align*}
\]
The desired polynomial has the following form:
\[ s^6 + s^5 6\omega_r + s^4 15\omega_r^2 + s^3 20\omega_r^3 + s^2 15\omega_r^4 + s\omega_r^5 + \omega_r^6, \]
where \( \omega_r \) – resonant frequency of the closed-loop system.

Through the comparison of relationships (8) and (9), the parameters of the analysed estimator can be calculated:

\[
\begin{align*}
k_1 &= 6\omega_r T_1 \\
k_2 &= -15\omega_r^3 T_1 T_c + \frac{T_1 T_{c1}}{T_2} + \frac{T_1 T_{c2}}{T_2} + 1 \\
k_3 &= 20\omega_r^3 T_1 T_c - k_1 \left( T_2 T_{c1} + T_3 T_{c2} + T_3 T_{c1} \right) \\
k_4 &= -15\omega_r^4 T_1 T_2 T_c + T_1 T_2 + T_3 - k_2 \left( T_2 + T_3 \right) \\
k_5 &= 15\omega_r^5 T_1 T_2 T_3 T_c - k_1 - k_3 \\
k_6 &= -\omega_r^6 T_1 T_2 T_3 T_c + k_2
\end{align*}
\]

By suitable selection of the value of the resonant frequency different dynamic responses of the observer can be obtained. The block diagram of the system Luenberger observer for three-inertia system is presented in Fig. 2.

Fig. 2. The schematic diagram of the LO for the three-mass system
Rys. 2. Schemat blokowy obserwatora Luenbergera dla układu trójmasowego

4. ROBUSTNESS ANALYSIS

In simulation tests the estimation quality of all system state variables is investigated. The input signals of the LO are taken from the closed-loop control structure presented in Fig. 3.
Although this structure is not designed for the three-inertia system, for small values of the control structure gains it allows to work the drive system properly. The transients of the motor speed and the electromagnetic torque are disturbed by the white noises which emulate the measurement noises. The input signals of the LO are presented in Fig. 4. Parameters of the analysed system are as follows: $T_1 = T_3 = 180$ ms, $T_2 = 46$ ms, $T_{c1} = T_{c2} = 1.3$ ms.

The system starts at time $t_1 = 0$ s with the reference value set to $\omega_{\text{ref}} = 0.25$. These values have been selected to avoid the work in the electromagnetic torque limit. Firstly the optimal gains of the LO are calculated for the assumed level of the measurement noises. In order to do that, the relationship between the resonant frequency of the LO and the estimation errors of all system state variables is determined. The estimation errors are calculated with the help of the following equation:
where: $N$ – total number of samples, $\nu$ – real variable, $\nu_e$ – estimating variable.

The relationship between the estimation error and the resonant frequency of the LO for every state variable of the system is presented in Fig. 5.

As is evident from Fig. 5, the smallest estimation errors exist in the motor speed (a). The level of the estimation error is similar in the gearbox and load speeds (c, e) as well in the estimates of the first and the second shaft torques (b, d). The load torque is reconstructed with the biggest error (f). All the presented characteristics possess the values of the resonant frequency for which estimation errors are the smallest (in the considered case it is approximately $184\, \text{s}^{-1}$). Then the transients of the real and estimation variables as well as the estimation errors for optimal value of the resonant frequency are presented in Fig. 6.

Big estimation errors appear in all state variables when the load torque is changed. The reduction of the difference between the real and the estimated variables during the transients can be obtained by increasing the values of the resonant frequency of the LO. However, it simultaneously increases the noise level of the estimated variables in the steady-state condition.

Next the system with changeable load side inertia has been considered. First the relationship between the optimal resonant frequency of the LO (guarantying the smallest estimation error) and different values of the inertia of the load machine is calculated. The obtained characteristics are presented in Fig. 7.

As can be concluded from the characteristics presented in Fig. 7, the smallest value of the resonant frequency exists for the nominal parameters of the plant. Changes of the time constant of the load machine require bigger value of the resonant frequency of the LO. This value is an compromise between the occurrence of the measurement noises in steady-state conditions and the reduction of the difference between the real and the estimated state variables (caused by parameters mismatch) during transients. Only the characteristic related to the load torque is different (Fig. 7f). In order to minimise the estimation errors for the bigger value of the time constant of the load machine, this variable requires the smaller gains of the LO.

The transients of the system for the three times bigger time constant of the load machine $T_3$ and the value of the LO coefficients set in order to minimise the error in the load machine speed are presented in Fig. 8. As can be observed from Fig. 8, the smallest estimation error (invisible in the presented scale) occurs in the transient of the motor speed. The errors in the transients of the first shaft torque and the gearbox speed are visible but they are relatively small. They appear during the start-up, reversal and the changes of the load torque. Relatively big errors arise in the transients of the second shaft torque and load speed. The biggest estimation errors are visible in the transient of the load torque.
They can be reduced by decreasing the value of the LO gains. However, it will increase the errors in the other of the state variables system.

Fig. 5. The relationship between the resonant frequency and the estimation errors: motor (a), gearbox (c) and load (e) speeds; the first shaft (b), second shaft (d) and load (f) torques

Rys. 5. Zależność pomiędzy pulsacją rezonansową obserwatora a błędami estymacji: prędkości silnika (a) przekładni (c) i maszyny roboczej (e) oraz momentów skrętnych w pierwszym (b) oraz drugim (d) wale i momentu obciążenia (f)
Fig. 6. The transients of the: motor (a), gearbox (c) and load (e) speeds as well as the first shaft (b), second shaft (d) and load (f) torques for nominal value of the system parameters

Rys. 6. Przebiegi prędkości silnika (a), przekładni (c) i obciążenia (e) oraz momentów skrętnych w pierwszym (b) i drugim (d) wale oraz momentu obciążenia dla znamionowych parametrów układu
Fig. 7. The relationship between the inertia of the load machine and the optimal value of the resonant frequency for all state variables: motor (a), gearbox (b) and load side (e) speeds and the first (b), the second (d) and the load torques (f)

Rys. 7. Zależność pomiędzy stałą czasową maszyny roboczej a optymalną wartością pulsacji rezonansowej dla następujących zmiennych stanu układu napędowego: prędkości silnika (a) przekładni (c) i maszyny roboczej (e) oraz momentów skrętnego w pierwszym (b) oraz drugim (d) wale i momentu obciążenia (f)
Fig. 8. The transients of the: motor (a), gearbox (c) and load (e) speeds as well as the first shaft (b), second shaft (d) and load (f) torques for increased value of the system parameters.

Rys. 8. Przebiegi prędkości silnika (a), przekładni (c) i obciążenia (e) oraz momentów skrętnych w pierwszym (b) i drugim (d) wale oraz momentu obciążenia dla zwiększonej wartości stałej czasowej maszyny roboczej.
5. CONCLUSIONS

In the paper the design and the robustness analysis of the LO for the three-inertia system has been carried out. The design procedure of the LO has been described in detail. The analytical equations which allow to set the LO coefficient are provided. It has been shown that for the system with measurement noises there is a specific value of the LO gain which ensures the smallest estimation error of the system state variables. In the case of the mismatch between the plant and the observer there is a need to increase the LO gain in order to ensure the smallest error of the reconstructed state variables. In this case all system state variables are reconstructed with relatively small errors, only the transient of the estimated load torque differs from the real one significantly.

REFERENCES